

Emergent Lorentz invariance with chiral fermions

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Abstract

We study renormalization group flows in strongly interacting field theories with fermions that correspond to transitions between a theory without Lorentz invariance at high energies down to a theory with approximate Lorentz symmetry in the infrared. Holographic description of the strong coupling is used. The emphasis is made on emergence of chiral fermions in the low-energy theory.

1 Introduction

Recent years have witnessed a revival of the idea [1, 2] that Lorentz invariance (LI), instead of being a fundamental property of Nature, may emerge only as an approximate symmetry of the low-energy physics. An important motivation to explore this possibility comes from the proposal by P. Hořava [3] that abandoning LI improves the ultraviolet behavior of gravity. In this proposal, departure from LI at large energies allows to postulate anisotropic (Lifshitz) scaling [4] which renders the gravity theory power-counting renormalizable. There is a version of Hořava–Lifshitz gravity [5, 6] which is viable and passes the existing experimental constraints [7]. On the phenomenological side, breaking of LI has been invoked in the construction of interesting long-distance modifications of gravity [8, 9, 10]. These provide a

playground for the analysis of possible deviations from general relativity and may be relevant for cosmology [11, 12].

Emergence of an effective Lorentz symmetry at low energies is a fairly common phenomenon in condensed matter physics, see e.g. [13, 14, 15, 16, 17, 18]. Also, it is not hard to construct particle physics models with this property. Indeed, the original works [1, 2] considered Yang–Mills theory and quantum electrodynamics with an addition of Lorentz violating terms of dimension four¹ in the Lagrangian and showed that the coefficients of those terms vanish along the renormalization group (RG) flow towards the infrared (IR). This result persists in the Lorentz violating extension of the full Standard Model [19]. Furthermore, a general argument [20, 21] shows that LI fixed points of RG flows are IR stable with respect to CPT -invariant deformations and thus have non-empty basin of attraction in the space of CPT -invariant non-relativistic theories. All theories belonging to this basin of attraction exhibit emergent LI². In two space-time dimensions it has been shown [22] that under broad assumptions interacting fixed points with *isotropic* scaling are necessarily LI.

However, this scenario faces an important challenge when confronted with the precision tests of LI in particle physics experiments [23, 24, 25]. In a weakly coupled theory the running of the coefficients in front of marginal Lorentz violating operators is logarithmic. If they are of order one³ at a certain ultraviolet (UV) scale Λ_* , say, of order Planck mass, they become only mildly suppressed at the experimentally accessible energies. These operators would, in particular, modify the dispersion relations of elementary particles producing species-dependent shifts of their maximal velocities [28]. Experimental constraints on this effect are very tight [24] and the logarithmic suppression is by far insufficient to satisfy them. Higher-dimension Lorentz violating operators are suppressed by Λ_* and are less problematic. Still, important bounds on their contributions into particles’ dispersion relations can be obtained from astrophysics [29, 30, 31, 32, 33].

There are two ways to address this challenge. One is to stay within perturbative regime and assume that the theory has another symmetry which entails LI as an accidental symmetry at low energies. Remarkably, the role of the extra symmetry can be played by non-relativistic

¹Lorentz violating terms of dimension two cannot be constructed from the fields of the theory and those of dimension three can be forbidden by imposing the discrete CPT symmetry.

²Of course, this does not imply that *any* non-relativistic theory becomes LI in the infrared; for example, there are RG flows terminating at fixed points with Lifshitz scaling.

³In principle, one could consider a setup where at tree level LI is an exact symmetry of the matter sector and is violated only in gravity [26]. Then, Lorentz violation is transmitted to matter only through loops with gravitons which are suppressed. However, the departure from the homogeneous Lifshitz scaling in the UV required by this approach can compromise the high-energy properties of the theory [27].

supersymmetry [34, 35]. Of course, in a realistic model the supersymmetry must be broken which leads to the generation of potentially dangerous Lorentz violating operators. However, if the scale of supersymmetry breaking is hierarchically lower than Λ_* , the coefficients of these operators are strongly suppressed and can be compatible with the experimental bounds. It should be mentioned though that it is unclear if the required version of supersymmetry is compatible with Lifshitz scaling at high energy [36].

Another approach was proposed in [21] (see also [37]) and is illustrated in Fig. 1. It is based on the observation that in strongly interacting theories the RG flow (understood in the Wilsonian sense) is fast and generally results in a power-law dependence of the couplings on the RG scale. Thus, if the Standard Model is embedded into a theory that passes through a strongly coupled RG evolution between the Lorentz violation scale Λ_* and some lower scale Λ_{IR} (still above the experimentally accessible energies), the coefficients of the Lorentz violating operators are suppressed by $(\Lambda_{IR}/\Lambda_*)^\alpha$ with $\alpha > 0$. The latter exponent is controlled by the dimension of the Least Irrelevant Lorentz Violating Operator (LILVO) [21]. If α and the hierarchy between the scales Λ_{IR} , Λ_* are large enough, the experimental constraints will be satisfied. Note that in this scenario it is sufficient that strong coupling occurs only in the matter sector and therefore is described, at least in principle, by the standard methods of quantum field theory. Gravity can remain weakly coupled at all energies.

It is worth stressing that this scenario implies that the whole SM sector passes through the regime of strong coupling. In other words, all SM fields must emerge as bound states of the strong dynamics with the compositeness scale set by Λ_{IR} . Construction of realistic models with this property is challenging. A useful tool is provided by the gauge/gravity (holographic) correspondence that enables to describe 4-dimensional strong dynamics in terms of weakly coupled theory in one dimension higher. This approach was used in [21] to implement the above mechanism for emergence of LI in a scalar toy model. The purpose of this work is to extend the analysis to theories with fermions. We focus on the case when the non-relativistic UV theory is invariant under spatial rotations, so fermions are defined as fields transforming in the spinor representation⁴ of $SO(3)$ (or $SO(d-1)$ in the general case of $(d-1)$ spatial dimensions).

Inclusion of fermions in the framework of emergent LI brings the following puzzle. The key role in the structure of the Standard Model is played by *chiral* fermions. However, the very definition of chirality relies on the existence of the Weyl representations of the Lorentz group. How chiral fermions can appear in a theory that fundamentally does not possess LI? We will see that the answer to this question relies on a simple kinematic property: an

⁴We shall assume that these fields obey anti-commutation relations.

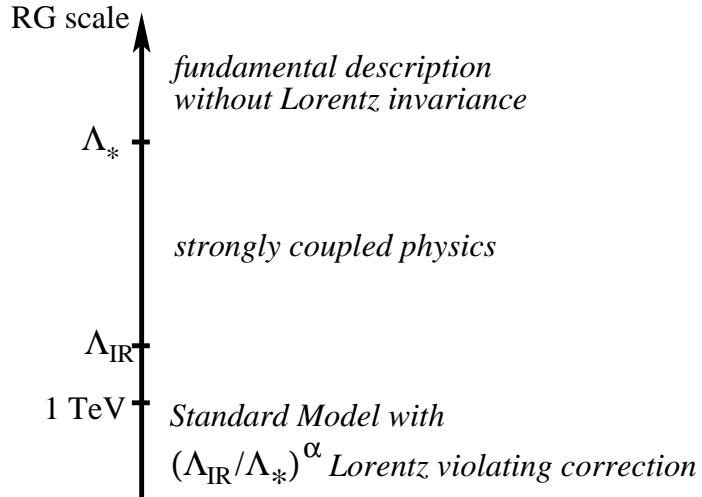


Figure 1: Various scales in the proposal for emergence of Lorentz invariance due to strong coupling. Above Λ_* the theory is essentially non-relativistic. The strongly coupled RG flow below Λ_* drives the theory towards a Lorentz invariant fixed point. A relevant deformation terminates the flow at Λ_{IR} where the theory enters into a confining phase and the Standard Model fields emerge as composite states.

$SO(3)$ spinor of the UV theory has two independent components which matches precisely the number of components of a four-dimensional Weyl spinor⁵. The degrees of freedom of the UV fermion are dressed by the strongly coupled RG evolution and at low energy constitute a chiral fermion. A necessary and sufficient requirement for this to happen is presence of a gapless mode in the low-energy theory. We will present holographic models of RG flows where this is indeed the case. These constructions will enable to compute explicitly the LV contributions in the low-energy theory and verify their power-law suppression.

The paper is organized as follows. In Sec. 2 we briefly review the holographic formalism for fermions in the standard relativistic setting. In Sec. 3 we consider a simplified model describing a non-relativistic fermion coupled to a strongly interacting relativistic sector. The latter is taken to be vector-like, i.e. it does not contain any massless chiral bound states. We show that this construction leads to appearance of a gapless chiral mode in the low-energy theory together with emergent LI. In Sec. 4 we turn to a setup modeling an RG flow from a theory with Lifshitz scaling in UV towards a LI infrared fixed point. We assume that the RG flow is stopped by the theory entering into a confining phase. We show that a suitable

⁵The match holds for any even-dimensional space-time.

pattern of confinement again leads to emergence of a chiral bound state. In Sec. 5 we discuss emergence of LI in theories with fermions in odd space-time dimensions. Section 6 is devoted to conclusions.

2 Holography for relativistic fermions

In this section we review the holographic correspondence for fermions in the LI case following [38, 39, 40, 41]. It establishes a relation between a strongly coupled conformal field theory (CFT) with large number of degrees of freedom in d dimensions and gravity in $(d + 1)$ -dimensional anti-de Sitter (AdS) space-time. The latter has the metric,

$$ds^2 = \left(\frac{l}{u}\right)^2 (-dt^2 + dx^i dx^i + du^2) , \quad i = 1, \dots, d-1, \quad (1)$$

where l is the AdS radius. A fermionic operator \mathcal{O}_ψ in the CFT corresponds to a fermion ψ in AdS. The action for a free fermion with mass M reads,

$$S = - \int d^{d+1}x \sqrt{|g|} i(\bar{\psi} \not{D} \psi - M \bar{\psi} \psi) + S_\partial , \quad (2)$$

where $\bar{\psi} = \psi^\dagger \Gamma^t$ and the Dirac operator is⁶

$$\not{D} = e_A^M \Gamma^A D_M , \quad D_M = \left(\partial_M + \frac{1}{2} \omega_{ABM} \Sigma^{AB} \right) , \quad \Sigma^{AB} = \frac{1}{4} [\Gamma^A, \Gamma^B] ; \quad (3)$$

Γ^A are the $(d + 1)$ -dimensional Dirac matrices. The boundary term S_∂ is needed to ensure the correct variational principle and will be specified shortly. The sign of the action is fixed by unitarity [41]. Choosing the diagonal vielbein and computing the corresponding spin connection,

$$e_A^M = (u/l) \delta_A^M , \quad \omega_{u\beta\mu} = \eta_{\beta\mu}/u , \quad (4)$$

with all other components of ω vanishing, one simplifies the expression for the Dirac operator,

$$\not{D} = \frac{u}{l} (\Gamma^u \partial_u + \Gamma^\mu \partial_\mu) - \frac{d}{2l} \Gamma^u . \quad (5)$$

⁶We use capital Latin letters from the beginning of the alphabet A, B, \dots for indices in the local Lorentz frame in $d + 1$ dimensions; capital letters from the middle of the alphabet M, N, \dots are used for $(d + 1)$ -dimensional space-time indices; Greek letters μ, ν, \dots denote indices in d dimensions spanned by t and x^i , $i = 1, \dots, d-1$ (as this space-time is flat, we do not distinguish the tangent-space indices and the indices in the local Lorentz frame).

To solve the Dirac equation

$$(\not{D} - M)\psi = 0 \quad (6)$$

one decomposes the spinor into eigenvectors with respect to Γ^u ,

$$\psi = \psi_+ + \psi_- , \quad \Gamma^u \psi_{\pm} = \pm \psi_{\pm} . \quad (7)$$

Performing the Fourier transform along the d flat dimensions, $\psi \propto e^{ip_\mu x^\mu}$, we obtain the general solution,

$$\psi_-(\vec{p}, u) = (pu)^{(d+1)/2} (\chi_1(\vec{p}) J_{Ml+1/2}(pu) + \chi_2(\vec{p}) Y_{Ml+1/2}(pu)) , \quad (8a)$$

$$\psi_+(\vec{p}, u) = (pu)^{(d+1)/2} \frac{ip_\mu \Gamma^\mu}{p} (\chi_1(\vec{p}) J_{Ml-1/2}(pu) + \chi_2(\vec{p}) Y_{Ml-1/2}(pu)) , \quad (8b)$$

where we have introduced $p \equiv \sqrt{-p_\mu p^\mu}$.

For $Ml > 1/2$ the mode associated with χ_2 is not normalizable. It is interpreted as a source for the operator \mathcal{O}_ψ in the CFT. More precisely, the source χ is defined by the relation

$$\chi = \lim_{u \rightarrow 0} (u/l)^{Ml-d/2} \psi_- . \quad (9)$$

The partition function,

$$Z[\chi] \equiv \int d\psi d\bar{\psi} e^{iS_{AdS}} , \quad (10)$$

with the integral taken over configurations satisfying the boundary condition⁷ (9) provides the generating functional for the correlators of \mathcal{O}_ψ . The CFT without sources corresponds to $\chi_2 = 0$. The dimension of \mathcal{O}_ψ is then read off from the small- u behavior of the component ψ_+ :

$$\psi_+ \propto \frac{ip_\mu \Gamma^\mu}{p} \chi_1(p) (pu)^{Ml+d/2} \implies \dim \mathcal{O}_\psi = Ml + \frac{d}{2} . \quad (11)$$

At $Ml < -1/2$ the situation is reversed. Now the non-normalizable mode is associated with χ_1 which dominantly contributes into the component ψ_+ . Thus, the latter is interpreted as a source, whereas ψ_- determines the dimension of the dual operator,

$$\dim \mathcal{O}'_\psi = -Ml + \frac{d}{2} . \quad (12)$$

In the window $-1/2 < Ml < 1/2$ both χ_1 and χ_2 modes are normalizable and the system admits two different quantizations [41]. In this paper we restrict to the range $Ml > -1/2$

⁷Generally, the integration in (10) should be performed over all fields present in the AdS theory, including the metric, subject to appropriate boundary conditions. For simplicity, we omit them in our discussion.

and choose the quantization with the source related to ψ_- by Eq. (9); the dimension of the operator is given by (11). Note that at $Ml = -1/2$ the expression (11) saturates the lower unitarity bound on the dimension of a spin-half operator in d -dimensional space-time [42, 43, 44].

The variation of the total action (2) must vanish on the solution. Integrating the bulk term by parts and setting the contribution at $u = \infty$ to zero we obtain,

$$\delta S = \lim_{\epsilon \rightarrow 0} i \int_{u=\epsilon} d^d x (\epsilon/l)^{-\frac{d-1}{2}} (\bar{\psi}_- \delta \psi_+ - \bar{\psi}_+ \delta \psi_-) + \delta S_{\partial} . \quad (13)$$

The component ψ_- is subject to the Dirichlet boundary condition (9), so its variation vanishes. However, $\delta \psi_+$ cannot be set to zero. To cancel this contribution one chooses [40]

$$S_{\partial} = - \lim_{\epsilon \rightarrow 0} i \int_{u=\epsilon} d^d x (\epsilon/l)^{-\frac{d-1}{2}} \bar{\psi}_- \psi_+ . \quad (14)$$

If the AdS space is bounded by a brane at finite ϵ , which corresponds to a finite UV cutoff in the dual CFT, the term (14) must be evaluated on this UV brane.

We assume that the CFT enters into a confining phase at low energy, so that it possesses a discrete spectrum of excitations, cf. [45]. This is realized by placing a brane⁸ at $u = L$ (referred to as “IR brane” below) and removing the region $u > L$. Physically, one interprets L^{-1} as the confining scale. We will consider two choices of boundary conditions on the IR brane, which we call \mathcal{B}_+ and \mathcal{B}_- . In the \mathcal{B}_+ case one imposes vanishing of ψ_+ on the brane,

$$\mathcal{B}_+ : \quad \psi_+|_{u=L} = 0 , \quad (15)$$

while the ψ_- component is arbitrary. In the case \mathcal{B}_- the situation is opposite,

$$\mathcal{B}_- : \quad \psi_-|_{u=L} = 0 , \quad \psi_+|_{u=L} \text{ --- arbitrary} . \quad (16)$$

The spectrum of masses is obtained by substituting (8) into Eq. (15) or (16) (recall that $\chi_2 = 0$ due to the boundary condition at $u \rightarrow 0$),

$$m_j^{(\pm)} = L^{-1} \mu_j^{(Ml \mp 1/2)} , \quad (17)$$

where $\mu_j^{(\nu)}$ is the j -th positive root of the Bessel function. Massless modes should be studied separately. A straightforward analysis shows that massless modes are absent in the case \mathcal{B}_+ . On the other hand, in the case \mathcal{B}_- there is a massless mode of the form,

$$\psi_+ = \chi_0(u/l)^{Ml+d/2} , \quad \psi_- = 0 , \quad (18)$$

⁸To solve the Einstein equations, the brane must have negative tension. This does not lead to pathologies if the gravitational field obeys suitable boundary conditions, see e.g. [46].

where the spinor χ_0 satisfies the equation $p_\mu \Gamma^\mu \chi_0 = 0$.

Henceforth we specify to the case of even d ; we will return to the odd- d case in Sec. 5. It is convenient to choose the $(d+1)$ -dimensional gamma matrices as

$$\Gamma^\mu = \gamma^\mu, \quad \mu = 0, \dots, d-1, \quad \Gamma^u = \gamma^{d+1}, \quad (19)$$

where γ^μ are d -dimensional Dirac matrices and γ^{d+1} is the chirality matrix — the d -dimensional analog of γ^5 . From (9) we see that the source χ and the corresponding operator \mathcal{O}_ψ belong to the Weyl representation of the d -dimensional Lorentz group. However, the spectrum of the bound states contains a chiral massless mode only in the case \mathcal{B}_- . In the case \mathcal{B}_+ all eigenmodes are massive and have both left and right components combining into the full Dirac representation. In this sense the dual low-energy theory arising in the \mathcal{B}_+ case is vector-like.

3 Chiral modes from non-relativistic fermions

We want to couple the (deformed) CFT of the previous section to a non-relativistic fermion and study the resulting RG flow. To this end we introduce a UV cutoff Λ_* and at this scale promote the source χ to an elementary dynamical field with Lorentz violating action S_χ . The total action at the scale Λ_* takes the form,

$$S = S_{CFT} + S_\chi + \Lambda_*^{-Ml} \int d^d x \bar{\chi} \mathcal{O}_\psi. \quad (20)$$

We will assume S_χ to be invariant under spatial rotations⁹. For concreteness, let us start from a simple choice,

$$S_\chi = -b \int d^d x i(\bar{\chi} \gamma^0 \partial_0 \chi + v \bar{\chi} \gamma^i \partial_i \chi), \quad (21)$$

where the velocity v of the χ -field is different from unity. The parameter b with dimension of length has been introduced to render the action dimensionless. Note that we have not included the Lorentz violating operator without derivatives $\bar{\chi} \gamma^0 \chi$ which is odd under the charge-conjugation (and CPT). For the moment we will assume these symmetries, though they are not essential for the present setup (see below).

On the AdS side the above setup is realized by cutting the space-time with a UV brane at $u = \Lambda_*^{-1}$ and supplementing the action (2) with the boundary term,

$$S_{UV} = -b \int_{u=\Lambda_*^{-1}} d^d x i(\bar{\psi}_- \gamma^0 \partial_0 \psi_- + v \bar{\psi}_- \gamma^i \partial_i \psi_-). \quad (22)$$

⁹It is worth noting, however, that the analysis of this section can be straightforwardly generalized to the case without spatial isotropy.

In this way we obtain a theory in the slice of AdS bounded by a UV and IR branes, cf. [46, 45]. Without loss of generality one can identify Λ_*^{-1} with the AdS radius l ; this is done from now on.

The canonical dimension of the elementary fermion χ is $(d-1)/2$. Recalling the dimension (11) of the operator \mathcal{O}_ψ we observe that the interaction between χ and \mathcal{O}_ψ in (20) has dimension $d + Ml - 1/2$. If $Ml > 1/2$ this interaction is irrelevant and the Lorentz violating sector simply decouples from the CFT in the infrared. Hence, no LI emerges in this case, similar to what happens in the scalar model at irrelevant coupling [21]. In what follows we restrict to the range

$$-1/2 < Ml < 1/2, \quad (23)$$

when the interaction is relevant and generates a strongly coupled RG flow. Furthermore, we will focus on the choice \mathcal{B}_+ of the IR boundary conditions. As discussed in the previous section, the CFT in this case does not possess any massless modes. Then the total system has a single gapless mode coming from the elementary fermion χ which, as we are going to see shortly, survives down to IR. On the other hand, in the case of boundary conditions \mathcal{B}_- the total number of gapless modes is 2 (one from the CFT and one from χ). It is straightforward to show that they pair and form a gap [45]. As our aim is to study the emergence of chirality, this case is not of interest to us.

From (2), (22) one derives the boundary condition on the UV brane,

$$[b(\gamma^0 \partial_0 + v \gamma^i \partial_i) \psi_- + \psi_+] \big|_{u=l} = 0. \quad (24)$$

The spectrum of eigenmodes is determined by imposing (24) together with the IR boundary condition (15) on the general bulk solution (8). This leads to the wavefunctions,

$$\psi_- = \chi_1 f_-(u), \quad \psi_+ = \frac{ip_\mu \gamma^\mu}{p} \chi_1 f_+(u), \quad (25)$$

with

$$f_\pm(u) = (pu)^{(d+1)/2} \left[J_{Ml \mp 1/2}(pu) - \frac{J_{Ml-1/2}(pL)}{Y_{Ml-1/2}(pL)} Y_{Ml \mp 1/2}(pu) \right] \quad (26)$$

and the spinor χ_1 obeying the relation,

$$\left[p_0 \gamma^0 \left(1 + p b \frac{f_-(l)}{f_+(l)} \right) + p_i \gamma^i \left(1 + v p b \frac{f_-(l)}{f_+(l)} \right) \right] \chi_1 = 0. \quad (27)$$

We are interested in the gapless mode whose d -dimensional momentum squared is much smaller than the confinement scale,

$$pl \ll pL \ll 1. \quad (28)$$

Upon expanding the ratio $f_-(l)/f_+(l)$ under these assumptions Eq. (27) simplifies,

$$\left[-\omega\gamma^0 \left(1 + (1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \right) + k_i \gamma^i \left(1 + v(1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \right) \right] \chi_1 = 0, \quad (29)$$

where we denoted $\omega \equiv -p_0$, $k_i \equiv p_i$. It is natural to take the parameters b and l to be of the same order as they both characterize the UV properties of the system. Then (29) has the same form as the standard relativistic equation for a Weyl spinor, up to corrections suppressed by a power of the small ratio (l/L) of the IR and UV cutoffs. Note that the maximal suppression achievable in this setup is $(l/L)^2$ which is reached at $Ml \approx -1/2$. Squaring the combination of the γ -matrices in (29) yields the dispersion relation,

$$\omega = \pm |\mathbf{k}| \left[1 + (v - 1)(1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \right]. \quad (30)$$

We see that the velocity of the mode is close to one for large hierarchy between the UV and IR scales. The equation for the amplitude of the positive-frequency mode then takes the form,

$$\Sigma_{\mathbf{k}} \chi_1 = 0, \quad \Sigma_{\mathbf{k}} \equiv -|\mathbf{k}| \gamma^0 + k_i \gamma^i. \quad (31)$$

Recalling that χ_1 also satisfies the chirality condition $\Gamma^u \chi = \gamma^{d+1} \chi_1 = -\chi_1$ (see (7), (19)), we conclude that it has $2^{d/2-2}$ independent components with the same structure as a Weyl spinor in d dimensions. In particular, in the case $d = 4$ it has a single component corresponding to the spin pointing along the direction of the spatial momentum.

The above derivation was carried under the assumptions (28). Using (30) one finds that the rightmost inequality is satisfied as long as the momentum of the mode does not exceed certain upper limit,

$$|\mathbf{k}| \ll \frac{1}{L\sqrt{|v-1|}} \left(\frac{L}{l} \right)^{1/2-Ml}. \quad (32)$$

This bound is parametrically larger than $1/L$, but smaller than $1/l$. To understand what happens at higher momenta we expand $f_{\pm}(l)$ for $pl \ll 1$ treating the product pL as a quantity of order one. This yields,

$$p b \frac{f_+(l)}{f_-(l)} = \frac{b}{l} \mathcal{J}(pL) (pl)^{1-2Ml},$$

where $\mathcal{J}(z)$ is a combination of Bessel functions with order-one coefficients. The eigenvalue

equation following from (27) can be cast into the form¹⁰,

$$\frac{(pL)^{1+2Ml}}{\mathcal{J}(pL)} = 2(v-1)\frac{b}{l}\left(\frac{l}{L}\right)^{1-2Ml} \mathbf{k}^2 L^2. \quad (33)$$

Note that the l.h.s. depends only on the LI combination $p = \sqrt{\omega^2 - \mathbf{k}^2}$. Inverting (33) we obtain the dispersion relation,

$$\omega^2 = \mathbf{k}^2 + \frac{1}{L^2} F\left[2(v-1)\frac{b}{l}\left(\frac{l}{L}\right)^{1-2Ml} \mathbf{k}^2 L^2\right], \quad (34)$$

where $F(z)$ is a dimensionless function. The dispersion relation at low momenta is obtained by expanding $F(z)$ in Taylor series. For the gapless mode the zeroth-order term vanishes and the linear term gives (30). It is instructive to consider the next term in the expansion which gives a contribution into the dispersion relation of order,

$$(v-1)^2(b/l)(l/L)^{-4Ml} l^2 \mathbf{k}^4. \quad (35)$$

For $Ml < 0$ the effective scale entering this contribution is parametrically higher than the UV scale l^{-1} . Thus, we conclude that the strongly coupled RG flow can suppress also the higher-order Lorentz violating corrections to the dispersion relation. For high $|\mathbf{k}|$ the Taylor expansion does not apply. The analysis then depends on the sign of $(v-1)$. If the velocity of the elementary fermion is superluminal, $v > 1$, Eq. (33) implies that at $|\mathbf{k}| \rightarrow \infty$ the combination pL approaches the first positive root of the function \mathcal{J} ,

$$pL \approx z_1, \quad \mathcal{J}(z_1) = 0.$$

In this regime the dispersion relation again becomes relativistic, but with a non-vanishing mass, cf. [21]. It is unclear whether this recovery of LI dispersion relation at high momenta is a peculiarity of holographic models, or is a more general property of emergent LI in strongly coupled systems with superluminal propagation. On the other hand, in the subluminal case $v < 1$ the deviation of the dispersion relation from the relativistic form grows with $|\mathbf{k}|$. Both in the sub- and superluminal cases it is easy to check that the spinor wavefunction satisfies (31) for all \mathbf{k} and hence describes a state with a fixed helicity.

Let us return to the case of low momenta satisfying (32) and write down the effective action for the gapless mode. This is obtained by substituting the wavefunctions (25) into

¹⁰In deriving (43) we have neglected terms on the l.h.s. suppressed by positive powers of pl . This is justified only for the gapless mode which we are interested in. For higher excitations the omitted terms are important.

the action composed of (2), (14) and (22). One finds that the bulk contribution vanishes and the action is given by the boundary terms on the UV brane¹¹,

$$\begin{aligned} S_{eff} &= \int d^d p \left[\frac{f_-^*(l)f_+(l)}{p} \bar{\chi}_1 p_\mu \gamma^\mu \chi_1 + b |f_-(l)|^2 \bar{\chi}_1 (-\omega \gamma^0 + v k_i \gamma^i) \chi_1 \right] \\ &= -i \int d^d x \left[\bar{\hat{\chi}}_1 \gamma^\mu \partial_\mu \hat{\chi}_1 + (1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \bar{\hat{\chi}}_1 (\gamma^0 \partial_0 + v \gamma^i \partial_i) \hat{\chi}_1 \right], \end{aligned} \quad (36)$$

where in the last equality we changed to the canonically normalized field

$$\hat{\chi}_1 = \chi_1 \sqrt{\frac{f_-^*(l)f_+(l)}{p}} \quad (37)$$

and switched back to the coordinate representation. We observe that the action consists of the LI piece describing a Weyl fermion and a suppressed Lorentz violating contribution¹². Variation of this action reproduces the equation of motion (24).

From the viewpoint of the dual field theory these results can be understood as follows. The coupling to the elementary fermion generates the RG flow towards the CFT corresponding to the alternative quantization of the bulk theory mentioned in Sec. 2. The latter contains a fermionic operator \mathcal{O}'_ψ with the dimension (12). The deviation from LI is governed at low energies by the irrelevant deformation of the CFT action,

$$\delta S_{CFT} \propto \int d^d x l^{1-2Ml} \bar{\mathcal{O}}'_\psi \gamma^0 \partial_0 \mathcal{O}'_\psi. \quad (38)$$

In the presence of confinement this deformation produces Lorentz violating corrections to the effective action of the bound states which depend on the ratio l/L precisely in the same way as in (36).

The analysis can be easily generalized to the case of an arbitrary bare action for the elementary fermion. Consider, for example, replacing (21) with

$$S'_\chi = b \int d^d x \bar{\chi} (-i \gamma^0 \partial_0 + \gamma^0 \Omega(-\Delta)) \chi, \quad (39)$$

where $\Omega(-\Delta)$ is a function of the spatial Laplacian $\Delta \equiv \partial_i \partial_i$. This action violates the charge conjugation and CPT . In the absence of coupling to the CFT it describes a non relativistic fermion with dispersion relation

$$\omega = \Omega(\mathbf{k}^2), \quad (40)$$

¹¹At this stage we do not impose the UV boundary condition (24).

¹²For a single field this contribution can be removed by a rescaling of the space or time coordinate. However, this will no longer be possible if the system contains several particle species.

whose spin degree of freedom is decoupled from the translational motion. In $d = 4$ this is a fermion with two states corresponding to the spin projections $\pm 1/2$ on an arbitrary axis. However, the situation changes qualitatively once we couple it to the bulk theory. The expression (39) translates into a boundary term on the UV brane,

$$S'_{UV} = b \int_{u=l} d^d x \bar{\psi}_- (-i\gamma^0 \partial_0 + \gamma^0 \Omega(-\Delta)) \psi_- , \quad (41)$$

which leads to the effective action for the light mode,

$$S'_{eff} = -i \int d^d x \left[\bar{\hat{\chi}}_1 \gamma^\mu \partial_\mu \hat{\chi}_1 + (1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \bar{\hat{\chi}}_1 \gamma^0 (\partial_0 + i\Omega(-\Delta)) \hat{\chi}_1 \right] . \quad (42)$$

The non-relativistic term in the action is again suppressed relative to the LI kinetic term induced from the bulk. The corresponding dispersion relation reads,

$$\omega = \pm |\mathbf{k}| \left[1 - (1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} \right] + \Omega(\mathbf{k}^2) (1 - 2Ml) \frac{b}{l} \left(\frac{l}{L} \right)^{1-2Ml} , \quad (43)$$

and the positive-frequency eigenspinor χ_1 satisfies the relation (31) implying that its polarization is aligned with the momentum. Note that in this setup CPT emerges¹³ together with LI.

4 Emergent chirality from Lifshitz flows

4.1 The domain wall geometry

Lifshitz space introduced in [47, 48] is a useful tool for holographic studies of non-relativistic theories. It possesses the isometry

$$t \mapsto \lambda^z t, \quad x_i \mapsto \lambda x_i, \quad i = 1, \dots, d-1 \quad (44)$$

$$u \mapsto \lambda u, \quad (45)$$

where λ is a scaling parameter. The AdS space-time is a special case of the Lifshitz geometry with $z = 1$. By extrapolating the ideas of holography to other values of z one expects the physics of Lifshitz space-time to capture the properties of strongly coupled d -dimensional non-relativistic theories invariant under the anisotropic scaling of time and space (44). Embeddings of Lifshitz solutions into supergravity and string theory have been discussed in [49, 50, 51, 52].

¹³If $\Omega(0)$ is non-zero, there is a small violation of CPT for modes with very low momentum, which can be interpreted as the presence of a chemical potential of order $\Omega(0)(l/L)^{1-2Ml}$.

To describe an RG flow from a Lifshitz UV fixed point to an IR theory with emergent LI, we need a geometry interpolating between Lifshitz space at small values of the holographic coordinate, $u \rightarrow 0$, and AdS at $u \rightarrow \infty$. Geometries with these properties can be obtained as solutions to the Einstein equations with matter represented by a massive vector field V_M . The action reads

$$S_V = \frac{1}{16\pi\kappa} \int d^{d+1}x \left(R - 2\Lambda_c - \frac{1}{4} F_{MN} F^{MN} - \frac{M_V^2}{2} V_N V^N \right), \quad (46)$$

where $\Lambda_c < 0$ is a cosmological constant and $F_{MN} = \partial_M V_N - \partial_N V_M$ is the field strength. A detailed study of this system was performed in [53] (see also [21]). It was shown that it has a family of solutions labeled by a parameter l_* , that are static and invariant under $SO(d-1)$ rotations of the spatial coordinates. The metric and the vector field have the form,

$$ds^2 = \left(\frac{l}{u} \right)^2 (-f^2(u) dt^2 + dx_i dx_i + g^2(u) du^2), \quad V_t = \frac{2}{M_V u} f(u) j(u), \quad V_i = V_u = 0, \quad (47)$$

where

$$l^2 = -\frac{d(d-1)}{2\Lambda_c}. \quad (48)$$

The functions f, g, j depend on u through the combination u/l_* and can be found numerically. They have the asymptotics

$$f = f_0(u/l_*)^{1-z}, \quad g = g_0, \quad j = j_0, \quad u \rightarrow 0, \quad (49a)$$

$$f = 1 + f_\infty(u/l_*)^{-2\alpha_V}, \quad g = 1 + g_\infty(u/l_*)^{-2\alpha_V}, \quad j = j_\infty(u/l_*)^{-\alpha_V}, \quad u \rightarrow \infty. \quad (49b)$$

Here $f_{0,\infty}, g_{0,\infty}, j_{0,\infty}$ are constants of order one and

$$\alpha_V = -\frac{d}{2} + \sqrt{\left(\frac{d}{2} - 1 \right)^2 + (M_V l)^2}. \quad (50)$$

For a given l_* the solution has the form of a domain wall interpolating between the Lifshitz and AdS spaces and centered at $u = l_*$. In Fig. 2 we plot the functions $j(u)$ and $g(u)$ for the case $d = 4$ and several values of the Lifshitz exponent. Numerical analysis shows that all three functions f, g, j are monotonically decreasing with u , so that the coefficients $f_\infty, g_\infty, j_\infty$ in (49b) are positive.

The domain wall solutions exist for the vector field masses in the range

$$d-1 \leq (M_V l)^2 \leq \frac{d(d-1)^2}{3d-4}. \quad (51)$$

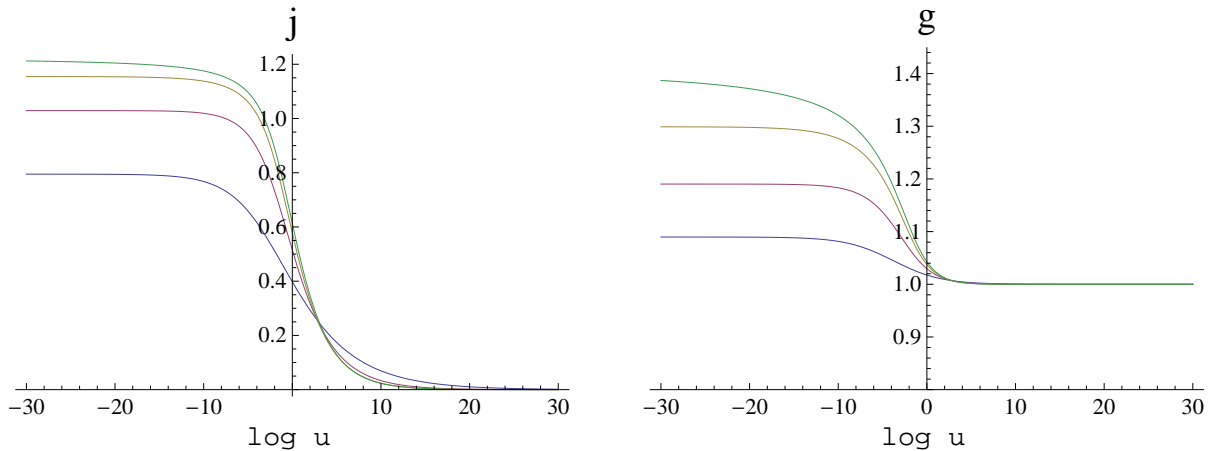


Figure 2: Functions $j(u)$ and $g(u)$ describing the domain wall solution interpolating between Lifshitz space at $u \rightarrow 0$ and AdS at $u \rightarrow \infty$ for several values of the Lifshitz exponent and $d = 4$. From top to bottom: $z = 3, 2.5, 2, 1.5$. The position l_* of the domain wall has been set to unity.

The Lifshitz exponent z is related to the space-time dimensionality d and the quantity $M_V l$. We will not need the explicit expression, which can be found in [21], and just note that for all values of the vector field mass satisfying (51) the exponent lies in the range $1 \leq z \leq d - 1$. Also, due to the upper bound (51) on $M_V l$ the exponent α_V is bounded from above and numerically turns out to be quite small for interesting values of d (for example, for $d = 3$ and $d = 4$ the upper limit on α_V is 0.13 and 0.35 respectively).

In the dual picture the position of the domain wall sets the energy scale $\Lambda_* = l_*^{-1}$ where the RG flow makes transition between the vicinities of the Lifshitz UV theory and the LI IR fixed point. The approach to the latter is governed by the vector operator \mathcal{O}_V^μ dual to the bulk vector field. The dimension of this operator in the IR theory is found using the standard rules of AdS/CFT,

$$\dim \mathcal{O}_V^\mu = d + \alpha_V . \quad (52)$$

Thus, the vector operator is irrelevant and the IR fixed point is attractive as long as $\alpha_V > 0$. From (50) we see that this condition coincides with the lower bound on the vector field mass in (51).

4.2 Fermions in the Lifshitz flow

We now add fermions to the above setup. We consider the action,

$$S = - \int d^{d+1}x \sqrt{|g|} i\bar{\psi} (\not{D} - \xi \not{D}^{(V)} - M) \psi, \quad (53)$$

where the Dirac operator and covariant derivatives are given by (3) and we have introduced a direct coupling of the fermion to the vector field via the operator¹⁴

$$\not{D}^{(V)} \equiv \frac{(M_V l)^2}{4} V_N V_M e_A^M \Gamma^A D^N. \quad (54)$$

It modifies the effective metric felt by the fermion in non-zero vector field background,

$$g_{MN} \mapsto g_{MN} - \xi \frac{(M_V l)^2}{4} V_N V_M. \quad (55)$$

The coupling constant ξ can be different for different fermion species and serves to probe species-dependent properties of the RG flow. Note that we have omitted the boundary term S_∂ . In contrast to Sec. 3, we will concentrate here on the behavior of the dual field theory without sources and correspondingly impose vanishing boundary conditions on ψ at $u \rightarrow 0$. In this case the boundary term vanishes together with its variation.

Taking a diagonal vielbein and computing the non-vanishing components of the spin connections,

$$\omega_{u00} = \frac{f'}{g} - \frac{f}{ug}, \quad \omega_{uij} = \frac{\delta_{ij}}{ug} \quad (56)$$

one obtains the expressions for the operators \not{D} and $\not{D}^{(V)}$ in the background (47),

$$\not{D} = \frac{u}{lg} \left[\partial_u + \frac{1}{2} \left(\frac{f'}{f} - \frac{d}{u} \right) \right] \Gamma^u + \frac{u}{lf} \Gamma^0 \partial_0 + \frac{u}{l} \Gamma^i \partial_i, \quad (57a)$$

$$\not{D}^{(V)} = -j^2 \left[\frac{u}{lf} \Gamma^0 \partial_0 + \frac{u}{2lg} \left(\frac{f'}{f} - \frac{1}{u} \right) \Gamma^u \right]. \quad (57b)$$

Decomposition of the spinor into eigenstates of Γ^u and Fourier transform along the (t, \mathbf{x}) coordinates yield the system of equations,

$$\pm (\partial_u + F_\pm(u)) \psi_\pm + ig (-\omega \gamma^0 + k_i \gamma^i) \psi_\mp = -i\omega G(u) \gamma^0 \psi_\mp, \quad (58)$$

where

$$F_\pm(u) = \frac{1}{2} \left[\frac{(1 + \xi j^2) f'}{f} - \frac{d + \xi j^2}{u} \right] \mp \frac{Mlg}{u}, \quad G(u) = g \left(1 - \frac{1 + \xi j^2}{f} \right). \quad (59)$$

¹⁴We do not consider a coupling without derivatives $\bar{\psi} V_N e_A^N \Gamma^A \psi$ which can be forbidden by the symmetry $V_N \mapsto -V_N$.

Note that the l.h.s. of (58) contains only LI operators, while all Lorentz violating contributions have been grouped on the r.h.s.

To obtain a discrete spectrum of excitations, we cut the space-time by an IR brane¹⁵ placed at $u = L \gg l_*$. As discussed in Sec. 2, in the relativistic case there is a massless fermion mode for the choice of boundary conditions \mathcal{B}_- (16). Let us show that this choice also gives rise to a gapless mode in the Lifshitz flow geometry. Consider first the limit of vanishing frequency and momentum. Setting $\omega = k_i = 0$ in (58) we obtain the solution,

$$\psi_+^{(0)} = \chi_0 \exp \int_u^L F_+(u') du' , \quad \psi_-^{(0)} = 0 , \quad (60)$$

where χ_0 is a constant chiral spinor satisfying¹⁶ $\gamma^{d+1} \chi_0 = \chi_0$. This solution behaves as

$$\psi_+^{(0)} \propto (u/L)^{Ml+d/2}$$

in the AdS region $u \gg l_*$. Taking into account the AdS measure $\int du u^{-d}$ entering into the mode normalization, we see that the normalization integral for (60) is saturated at $u \sim L$ if $Ml > -1/2$. In other words, the mode (60) is suppressed in the Lifshitz part of the space-time. This suggests the following strategy to solve Eqs. (58) for non-zero momenta. As the zeroth approximation one takes (60) together with the LI dispersion relation¹⁷ $\omega = |\mathbf{k}|$. This yields the constraint on the spinor amplitude,

$$\Sigma_{\mathbf{k}} \chi_0 = 0 , \quad (61)$$

where $\Sigma_{\mathbf{k}}$ has been defined in (31). Thus χ_0 corresponds to a fixed helicity. The Lorentz violating contributions are then treated as small corrections.

We write,

$$\psi_+ = \psi_+^{(0)} + \psi_+^{(1)} , \quad \psi_- = \psi_-^{(1)} , \quad \omega = |\mathbf{k}| + \omega^{(1)} . \quad (62)$$

Substituting into (58) and keeping only terms linear in the perturbations we obtain,

$$(\partial_u + F_+) \psi_+^{(1)} + ig \Sigma_{\mathbf{k}} \psi_-^{(1)} = 0 , \quad (63a)$$

$$-(\partial_u + F_-) \psi_-^{(1)} + ig \Sigma_{\mathbf{k}} \psi_+^{(1)} - ig \omega^{(1)} \gamma^0 \psi_+^{(0)} = -i |\mathbf{k}| G \gamma^0 \psi_+^{(0)} . \quad (63b)$$

One multiplies the first equation by $\Sigma_{\mathbf{k}}$ and uses the identity $\Sigma_{\mathbf{k}}^2 = 0$ to eliminate the second term. The remaining equation has the solution,

$$\Sigma_{\mathbf{k}} \psi_+^{(1)} = \tilde{\chi} \exp \int_u^L F_+(u') du' , \quad (64)$$

¹⁵The brane energy-momentum tensor required for a static solution of Einstein's equations is composed of a negative tension and a contribution satisfying the null energy condition [21].

¹⁶Recall that for our choice of the bulk Γ -matrices $\Gamma^u = \gamma^{d+1}$, see (19).

¹⁷We focus on the positive-frequency mode; for $\omega = -|\mathbf{k}|$ the analysis is similar.

where the constant spinor $\tilde{\chi}$ satisfies the relations

$$\gamma^{d+1}\tilde{\chi} = -\tilde{\chi} , \quad \Sigma_{\mathbf{k}}\tilde{\chi} = 0 . \quad (65)$$

Substitution of (64) into (63b) yields the solution for $\psi_-^{(1)}$,

$$\psi_-^{(1)}(u) = \tilde{\psi}_-^{(1)}(u) \exp \int_u^L F_-(u') du' , \quad (66)$$

where

$$\tilde{\psi}_-^{(1)}(u) = \int_0^u du' [i(|\mathbf{k}|G(u') - \omega^{(1)}g(u'))\gamma^0\chi_0 + ig(u')\tilde{\chi}] \exp \left[-2Ml \int_{u'}^L \frac{g(u'')du''}{u''} \right] . \quad (67)$$

In deriving the last expression we have used $F_+ - F_- = -2Mlg/u$. Note that the outer integral in (67) is taken from $u = 0$ to ensure vanishing of the field on the UV boundary.

We now impose the boundary condition \mathcal{B}_- on the IR brane,

$$\tilde{\psi}_-(L) = 0 . \quad (68)$$

Multiplying (67) successively by $\Sigma_{\mathbf{k}}$ and $\tilde{\Sigma}_{\mathbf{k}} \equiv |\mathbf{k}|\gamma^0 + k_i\gamma^i$ and using the relations,

$$\Sigma_{\mathbf{k}}\gamma^0\chi_0 = 2|\mathbf{k}|\chi_0 , \quad \tilde{\Sigma}_{\mathbf{k}}\gamma^0\chi_0 = 0 , \quad \tilde{\Sigma}_{\mathbf{k}}\tilde{\chi} = 2|\mathbf{k}|\gamma^0\tilde{\chi} . \quad (69)$$

we find that the spinor $\tilde{\chi}$ must vanish,

$$\tilde{\chi} = 0 , \quad (70)$$

whereas the correction to the dispersion relation reads,

$$\omega^{(1)} = |\mathbf{k}| \frac{\int_0^L du G(u) \exp \left[-2Ml \int_u^L \frac{g(u')du'}{u'} \right]}{\int_0^L du g(u) \exp \left[-2Ml \int_u^L \frac{g(u')du'}{u'} \right]} . \quad (71)$$

Finally, the correction $\psi_+^{(1)}$ is found from (63a),

$$\psi_+^{(1)} = \tilde{\psi}_+^{(1)}(u) \exp \int_u^L F_+(u') du' \quad (72)$$

with

$$\tilde{\psi}_+^{(1)}(u) = -i \int_u^L du' g(u') \Sigma_{\mathbf{k}} \tilde{\psi}_-^{(1)}(u') \exp \left[2Ml \int_{u'}^L \frac{g(u'')du''}{u''} \right] . \quad (73)$$

Note that the corrections preserve the property that the fermion wavefunction on the IR brane $\psi_+(L)$ satisfies $\Sigma_{\mathbf{k}}\psi_+(L) = 0$ implying that it describes a particle of fixed helicity.

Our next task is to infer the scaling of $\omega^{(1)}$ with the IR cutoff L^{-1} . We notice that for $Ml > -1/2$ the integral in the denominator of (71) is saturated in the AdS region where $g(u)$ can be replaced by one. This yields,

$$\int_0^L du g(u) \exp \left[-2Ml \int_u^L \frac{g(u') du'}{u'} \right] = \frac{L}{1+2Ml} . \quad (74)$$

To evaluate the numerator we recall that the functions f, g, j depend on u only through the combination $z \equiv u/l_*$ (see Sec. 4.1). Switching to the latter variable inside the integrals we have,

$$\begin{aligned} \int_0^L du G(u) \exp \left[-2Ml \int_u^L \frac{g(u') du'}{u'} \right] &= l_* \int_0^{L/l_*} dz G(z) \exp \left[-2Ml \int_z^{L/l_*} \frac{g(z') dz'}{z'} \right] \\ &= l_* \left\{ \int_0^{z_0} dz G(z) \exp \left[-2Ml \int_z^{z_0} \frac{g(z') dz'}{z'} - 2Ml \int_{z_0}^{L/l_*} \frac{g(z') dz'}{z'} \right] \right. \\ &\quad \left. + \int_{z_0}^{L/l_*} dz G(z) \exp \left[-2Ml \int_z^{L/l_*} \frac{g(z') dz'}{z'} \right] \right\} \\ &= l_* \left\{ a_1 \left(\frac{l_*}{L} \right)^{2\alpha_V - 1} + a_2 \left(\frac{l_*}{L} \right)^{2Ml} \right\} , \end{aligned} \quad (75)$$

where the coefficients

$$a_1 = \frac{f_\infty - \xi j_\infty^2}{2Ml - 2\alpha_V + 1} , \quad (76a)$$

$$a_2 = z_0^{2Ml} \int_0^{z_0} dz G(z) \exp \left[-2Ml \int_z^{z_0} \frac{g(z') dz'}{z'} \right] - \frac{(f_\infty - \xi j_\infty^2) z_0^{2Ml - 2\alpha_V + 1}}{2Ml - 2\alpha_V + 1} \quad (76b)$$

are independent of l_* and L . On the other hand, they depend on the direct coupling ξ between the fermion and the vector field and thus can be different for different fermion species. In the second line of (75) we divided the integral in two parts with the intermediate point z_0 lying in the range $1 \ll z_0 \ll L/l_*$ and then used the asymptotic form of the metric functions (49b) at $z > z_0$. Of course, the expression in the last line of (75) is independent of the choice of z_0 . Combining (75) and (74) we obtain

$$\omega^{(1)} = |\mathbf{k}| \left[a_1 (1+2Ml) \left(\frac{l_*}{L} \right)^{2\alpha_V} + a_2 (1+2Ml) \left(\frac{l_*}{L} \right)^{1+2Ml} \right] . \quad (77)$$

This gives power-suppressed correction to the fermion velocity as long as $\alpha_V > 0$, $Ml > -1/2$. To sum up, similar to the model of Sec. 3, we have found that the gapless fermion mode has an almost LI dispersion relation with the number and structure of independent degrees

of freedom matching those of a Weyl fermion in d dimensions (a single helicity component for $d = 4$). Clearly, this mode is described by an approximately LI effective action with the fermion field transforming in the chiral representation of the Lorentz group.

The form of corrections (77) to the relativistic dispersion relation has a natural interpretation in the dual field theory. At the IR fixed point the theory is described by a CFT containing a spinor operator \mathcal{O}_ψ and a vector operator \mathcal{O}_V^μ with dimensions (11) and (52). The RG flow corresponds to deforming the CFT action by an irrelevant Lorentz violating perturbation,

$$\delta S_{CFT} = \int d^d x (c_1 l_*^{\alpha_V} \mathcal{O}_V^0 + c_2 l_*^{1+2Ml} \bar{\mathcal{O}}_\psi \gamma^0 \partial_0 \mathcal{O}_\psi) , \quad (78)$$

where $c_{1,2}$ are some dimensionless constants. These two terms generate the two contributions in (77) after the theory enters into the confining phase. Note that a double insertion of the deformation \mathcal{O}_V^0 is required to affect the fermion dispersion relation, as in the unperturbed CFT the three-point function $\mathcal{O}_\psi \bar{\mathcal{O}}_\psi \mathcal{O}_V$ vanishes due to the symmetry of the bulk action under $V_M \mapsto -V_M$.

Finally, let us comment on the domain of validity of the calculation leading to (77). A necessary requirement is that the correction (72) to the fermion wavefunction is smaller than the wavefunction at the zeroth order (60). From the estimates

$$\psi_+^{(1)} / \psi_+^{(0)} \sim k L \tilde{\psi}_-^{(0)} / \chi_0 \sim k L^2 \omega^{(1)} \quad (79)$$

one obtains the condition

$$|\mathbf{k}| \ll L^{-1} \min \{ (L/l_*)^{\alpha_V}, (L/l_*)^{1/2+Ml} \} . \quad (80)$$

Alternatively, the same condition can be derived from the requirement that the correction to the gapless mode energy $\omega^{(1)}$ is smaller than the splitting between the energies of the massless mode and the next eigenstate of the unperturbed relativistic Dirac operator. The latter eigenstate has relativistic dispersion relation with the mass of order $1/L$, so the splitting is of order $(|\mathbf{k}|L^2)^{-1}$. The upper limit (80) on the momentum is parametrically larger than the gap $1/L$ between the gapless mode and the next bound state. It may even seem that the perturbative expansion can be pushed to momenta higher than the Lorentz violation scale $1/l_*$ if $\alpha_A > 1$ and $Ml > 1/2$. However, we expect that the structure of higher order terms in the expansion will restrict its domain of validity to $|\mathbf{k}| \ll 1/l_*$, similar to the case of scalar theory [21]. Anyway, this issue is irrelevant for the setup studied in this section: as pointed in Sec. 4.1, for the interesting values of space-time dimensionality the exponent α_V is smaller than 1.

5 RG flows with fermions in odd dimensions

In this section we consider the holographic RG flow in the case when the space-time dimensionality d of the dual field theory is odd. This affects the form of the bulk Γ -matrices. Instead of (19) we choose,

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \mu = 0, \dots, d-1, \quad \Gamma^u = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (81)$$

where γ^μ are the d -dimensional Dirac matrices. Then the analysis of Secs. 3, 4 goes through with minor changes. One concludes that the two-brane model with the action (22) or (41) on the UV brane and \mathcal{B}_+ boundary conditions on the IR brane gives rise to a gapless fermion mode with almost LI dispersion relation. The same holds for the case of Lifshitz domain wall with \mathcal{B}_- boundary conditions in the IR. Of course, there is no notion of chirality in odd dimensions and the fermions transform in the Dirac representation of $SO(d-1, 1)$. The absence of the gap is protected by spatial parity which forbids a LI mass term for fermions in odd dimensions (see e.g. [54]). Indeed, under the reflection of a single axis a fermion in odd dimensions transforms as,

$$\chi(x^0, x^1, x^2, \dots) \mapsto \gamma^1 \chi(x^0, -x^1, x^2, \dots).$$

It is straightforward to check that a mass term $m\bar{\chi}\chi$ changes sign under this transformation. On the other hand, the original bulk action (2) or (53) respects parity which in $(d+1)$ even dimensions can be defined as

$$\psi(x^0, x^1, x^2, \dots) \mapsto \Gamma^{d+2} \Gamma^1 \psi(x^0, -x^1, x^2, \dots),$$

where $\Gamma^{d+2} = i\Gamma^0 \Gamma^1 \dots \Gamma^{d-1} \Gamma^u$.

A new feature that appears in the case of odd d is a possible mismatch between the number of degrees of freedom of non-relativistic and relativistic fermions. Namely, the spinor representation of the group $SO(d-1)$ of spatial rotations can be decomposed into the left and right parts, each having dimension $2^{(d-3)/2}$, which is thus the minimal number of components of a non-relativistic spinor. This does not fit into the Dirac representation of $SO(d-1, 1)$ which has $2^{(d-1)/2}$ components. Therefore, one does not expect emergence of LI in systems with minimal non-relativistic fermions. Let us analyze this question explicitly. We take $d = 3$ and consider the holographic setup with two branes similar to that studied in Sec. 3. First, we describe the model from the 3-dimensional viewpoint. The spinor operator \mathcal{O}_ψ of the dual CFT can be decomposed into the upper and lower components,

$$\mathcal{O}_\psi = \begin{pmatrix} \mathcal{O}_U \\ \mathcal{O}_D \end{pmatrix}. \quad (82)$$

Taking the γ -matrices in the form,

$$\gamma^0 = i\sigma_3, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_2, \quad (83)$$

where σ_i are the Pauli matrices, we observe that \mathcal{O}_U and \mathcal{O}_D do not mix under purely spatial rotations. Hence, it is consistent with spatial isotropy to couple only one of them, say \mathcal{O}_U , to an external dynamical field. This leads to the action,

$$S = S_{CFT} + l^{Ml} \int d^3x \chi_U^\dagger \mathcal{O}_U + b \int d^3x \chi_U^\dagger (i\partial_0 - \Omega(-\Delta)) \chi_U, \quad (84)$$

where χ_U is a single-component minimal $SO(2)$ spinor and $\Omega(-\Delta)$ is an arbitrary function of the spatial Laplacian. Note that this action breaks the discrete symmetries C , P , T and CPT as all of them interchange \mathcal{O}_U and \mathcal{O}_D .

The holographic description of this model is obtained by making only the upper component of the bulk spinor ψ_- on the UV brane dynamical with the action,

$$S_{UV} = b \int_{u=l} d^3x \psi_{-,U}^\dagger (i\partial_0 - \Omega(-\Delta)) \psi_{-,U}, \quad (85)$$

whereas imposing the Dirichlet boundary condition on the lower component,

$$\psi_{-,D}|_{u=l} = 0. \quad (86)$$

From (85), (2) we obtain the second boundary condition on the UV brane,

$$[b(i\partial_0 - \Omega(-\Delta))\psi_{-,U} + \psi_{+,U}]|_{u=l} = 0. \quad (87)$$

Combining with \mathcal{B}_+ boundary conditions on the IR brane and substituting the bulk solution (25), (26) into (86), (87) yields,

$$[mb(\omega - \Omega(k^2))f_-(l) + \omega f_+(l)]\chi_{1,U} = 0. \quad (88)$$

Expansion of the functions $f_\pm(l)$ at small argument gives the dispersion relation of the gapless mode,

$$\omega = \Omega(k^2)(1 - 2Ml)\frac{b}{l}\left(\frac{l}{L}\right)^{1-2Ml}. \quad (89)$$

Clearly, this is not relativistic. In the infrared limit $l/L \rightarrow 0$ the dispersion relation becomes degenerate, $\omega = 0$, implying that, the excitation of this mode does not cost any energy. In other words, the system contains infinite number of states with zero energy different from the vacuum. This situation appears rather pathological. It is excluded if we impose from the beginning one of the symmetries C , P , T or CPT .

6 Conclusions

In this paper we have studied emergence of LI in strongly coupled field theories with fermions. We considered two models. In the first model a strongly interacting relativistic CFT is coupled at a high energy scale to an elementary fermion with non-relativistic action. This generates an RG flow exhibiting LI in the IR. The second setup describes a field theory flowing from a UV fixed point with Lifshitz scaling to a LI IR fixed point. In both cases we assumed that the theories enter into a confinement phase at low energies, which gives rise to a discrete spectrum of excitations. We used the holographic duality to describe the strong coupling.

In the main part of the paper we focused on field theories living in even-dimensional space-time and analyzed how the notion of chirality arises together with emergent LI. We have found that if the theory possesses low-lying fermionic excitations, the latter are described by approximately LI effective actions with the fermion wavefunction transforming in the Weyl representation of the Lorentz group. The Lorentz violating corrections to the effective action are power-law suppressed by the ratio between the IR confinement scale and the UV scale of Lorentz violation. The exponent in the power-law is related to the dimension of the Least Irrelevant Lorentz Violating Operator (LILVO) in the IR theory, in complete analogy with the scalar case studied in [21]. A kinematic property that appears essential for such behavior is the coincidence between the number of independent components of the non-relativistic fermion operator in the UV theory and the relativistic Weyl spinor.

In the last section we have addressed the case of odd space-time dimensionality. We observed that the setups preserving the kinematic match mentioned above still lead to low-energy theories with approximately relativistic fermions which are protected from acquiring a mass by spatial parity. We also considered a model that violates the above match and showed that in this case the low-lying fermion mode does not exhibit LI. We pointed out that this situation is excluded if the theory respects one of the discrete symmetries C , P , T or CPT . This is one more manifestation of the deep role played by these symmetries in the phenomenon of emergent LI.

From the phenomenological perspective, our work represents a step towards implementing the idea of emergent LI in a realistic particle physics setting. Of course, an important open issue on this way is inclusion of gauge fields. As discussed in [21], the holographic description of strongly coupled RG flows with gauge fields will presumably require considering a more complicated bulk sector containing an interaction of the gauge fields with a dilaton. More generally, one can ask whether the gauge symmetry can be an emergent property appearing

together with emergent LI.

Another interesting question concerns the minimal value of LV couplings predicted in this framework. We have demonstrated that these couplings are suppressed by a power-law factor $(\Lambda_{IR}/\Lambda_*)^\alpha$, where the ratio Λ_{IR}/Λ_* characterizes the “duration” of the RG flow leading to the emergence of LI. Clearly, this duration cannot be arbitrarily large. In the models inspired by quantum gravity the natural value for Λ_* is at or below the Planck scale. In particular, in the context of the Horava’s proposal, Λ_* should be less than 10^{15} GeV [55, 6]. On the other hand, the scale Λ_{IR} is bounded below by particle physics experiments. As mentioned in the Introduction, Λ_{IR} sets the scale of compositeness for the SM fields. Direct searches for excited fermionic states¹⁸ put a lower bound on Λ_{IR} at the level of a few TeV [56, 57]. Taking for the estimate $\Lambda_{IR} \gtrsim 10$ TeV, we find that the duration of the RG flow is bounded from below by $\Lambda_{IR}/\Lambda_* \gtrsim 10^{-11}$. In all holographic examples considered in Ref. [21] and this paper the exponent α in the suppression has been found to be smaller than 2. Thus, the minimal size of LV operators predicted by these models is of order 10^{-22} , which is marginally compatible with the existing bounds [24]. In this light it will be important to understand whether the condition $\alpha < 2$ is universal or there exist RG flows with emergent LI that avoid it.

To conclude, the scenario of emergent LI provides an interesting interplay between high-energy particle physics and precision tests of relativity. An improvement of the experimental lower limits on the compositeness scale of SM fields and / or tightening of the constraints on LV parameters will be able to test this idea or rule it out.

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¹⁸More stringent limits can come from the physics of flavor, but they are model dependent.

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